Journal of
Mechanical
Science and
Technology

Journal of Mechanical Science and Technology 21 (2007) 1162~1171

Improved Moment-Based Quadrature Rule and Its Application to Reliability-Based Design Optimization

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(Manuscript Received November 14, 2006; Revised May 15, 2007; Accepted May 15, 2007)

Abstract

A moment method known as the fourth moment method can perform reliability analysis without optimization using the first four statistical moments. Numerical integration is used to calculate the statistical moments, where a moment-based quadrature rule can be used for integration nodes and weights. However, the moment-based quadrature rule has to solve a system of linear equations that can be numerically unstable. Considering this point, an improved moment-based quadrature rule is proposed and is applied to reliability-based design optimization. Finally, the moment-based RBDO is applied to numerical examples with a variety of random variables and target reliability indexes. From the numerical results, the performance of the improved moment-based quadrature rule can be confirmed and several guidelines are given for the moment-based RBDO.

Keywords: Moment-based quadrature rule; Moment method; Reliability-based design optimization

1. Introduction

Reliability analysis has been used to consider uncertainties existing in the real world. Typical methods for reliability analysis are Monte Carlo simulation (MCS) (Madsen et al., 1986), the first order reliability method (FORM) (Hasofer and Lind, 1974), the second order reliability method (SORM) (Breitung, 1984; Kiureghian et al., 1987), and a moment method (Zhao and Ono, 2001; Seo and Kwak, 2002). Of these methods, the moment method has an advantage in that it does not require optimization for reliability analysis. Instead of performing optimization, the moment method requires the first four statistical moments of a performance function, where the statistical moments are calculated by numerical integration.

For the numerical integration, integration nodes and weights should be known. Although integration nodes and weights for a standard normal distribution are generally known, those for other distributions should be calculated. To do this, Zhao and Ono (2000) proposed using an inverse Rosenblatt transformation for non-normal distributions, where integration nodes were only changed. As a method for considering both integration nodes and weights, Rahman and Xu (2004) proposed a moment-based quadrature rule (MBQR), in which a system of linear equations should be solved in order to obtain integration nodes. However, a drawback of the moment-based quadrature rule is that the system of linear equations may be singular, which can reduce the accuracy of the integration nodes and weights (Youn et al., 2006). Considering this point, an improved moment-based quadrature rule (IMBQR) that improves numerical stability is proposed.

As an application of the IMBQR, the IMBQR is

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applied to RBDO. This is done because previous research on RBDO has focused on methods based on FORM, such as the reliability index approach (RIA) (Yu et al., 1998) and the performance measure approach (PMA) (Tu and Choi, 1999; Youn et al., 2003). Furthermore, there has been almost no research done on applying the moment method to RBDO.

To do this, problems related to combining the moment method with RBDO are initially understood and then solutions to these are given. Based on the solutions, a moment-based RBDO is constructed. Using numerical examples with a variety of random variables and target reliability indexes, the performance of the IMBQR is confirmed and comments regarding the moment-based RBDO are given.

2. A moment method

2.1 Calculation of integration nodes and weights

A moment method for reliability analysis requires the first four statistical moments of a performance function and a Pearson system. To calculate the statistical moments, numerical integration is used. A typical formulation of the statistical moments for the Pearson system is as follows:

$$\begin{cases} \mu_g = \sum_{i_1=1}^{m} w_{i_1} \cdots \sum_{i_n=1}^{m} w_{i_n} g\left[T^{-1}(x_{i_1},\ldots,x_{i_n})\right] \\ \sigma_g = \left[\sum_{i_1=1}^{m} w_{i_1} \cdots \sum_{i_n=1}^{m} w_{i_n} \left(g\left[T^{-1}(x_{i_1},\ldots,x_{i_n})\right]-\mu_g\right)^2\right]^{\frac{1}{2}} \\ \sqrt{\beta_{1g}} = \left[\sum_{i_1=1}^{m} w_{i_1} \cdots \sum_{i_n=1}^{m} w_{i_n} \left(g\left[T^{-1}(x_{i_1},\ldots,x_{i_n})\right]-\mu_g\right)^3\right] \Big/ \sigma_g^3 \\ \beta_{2g} = \left[\sum_{i_1=1}^{m} w_{i_1} \cdots \sum_{i_n=1}^{m} w_{i_n} \left(g\left[T^{-1}(x_{i_1},\ldots,x_{i_n})\right]-\mu_g\right)^4\right] \Big/ \sigma_g^4 \end{cases}, \quad (1)$$

where w_i is the integration weights, x_i is the ith component of an integration node, m is the number of integration nodes, and T^{-1} is the inverse Rosenblatt transformation to deal with non-normal distributions.

In the case of a standard normal distribution, integration nodes and weights can be easily determined (Abramowitz and Stegun, 1972). However, when a non-normal distribution is handled, using the inverse Rosenblatt transformation may cause errors in the statistical moments because the integration nodes are only changed and the integration weights are identical to those of a standard normal distribution (Zhao and Ono, 2000).

This problem can be solved by using a moment-based quadrature rule (MBQR) (Rahman and Xu, 2004), where integration nodes and weights are obtained by solving a linear system equation that requires the statistical information of the input parameters. The linear relationship is made between low- and high-order moments of the input random variables, as shown in Eq. (2).

$$\begin{bmatrix} \mu_{j,n-1} & -\mu_{j,n-2} & \mu_{j,n-3} & \cdots & (-1)^{n-1}\mu_{j,0} \\ \mu_{j,n} & -\mu_{j,n-1} & \mu_{j,n-2} & \cdots & (-1)^{n-1}\mu_{j,1} \\ \mu_{j,n+1} & -\mu_{j,n} & \mu_{j,n-1} & \cdots & (-1)^{n-1}\mu_{j,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{j,2n-2} & -\mu_{j,2n-3} & \mu_{j,2n-4} & \cdots & (-1)^{n-1}\mu_{j,n-1} \end{bmatrix} \begin{bmatrix} r_{j,1} \\ r_{j,2} \\ r_{j,3} \\ \vdots \\ r_{j,n} \end{bmatrix} = \begin{bmatrix} \mu_{j,n} \\ \mu_{j,n+1} \\ \mu_{j,n+2} \\ \vdots \\ \mu_{j,2n-1} \end{bmatrix} \quad (2)$$

Here, $\mu_{j,n}$ represents the n^{th} raw moment considering the j^{th} input random variable. However, the system of linear equations may be singular when the number of integration nodes increases or the random variable has a large mean or a small coefficient of variance.

This may be a significant problem when the MBQR is applied to RBDO because the system of linear equations should be solved every time a design point is changed during the process of optimization. Furthermore, if the system of linear equations is singular even once during the process of optimization, optimization may fail to find an optimum. Therefore, it is important to construct the system of linear equations to be well-conditioned.

2.2 An improved moment-based quadrature rule

To apply the MBQR to RBDO, it is desirable that the case in which singularity occurs should be reduced by as much as possible. In general, the singularity of the system of linear equations can be determined by the condition number; the larger the condition number is, the larger the error of the system of linear equations is.

To reduce the condition number by as much as possible, an improved moment-based quadrature rule (IMBQR) is proposed, in which a constant that can control the condition number is introduced. To construct the IMBQR with n integration nodes $x_{j,i}, i=1,\ldots,n$ in the direction of the x_j coordinate, if a constant c is introduced and y_j is substituted for $x_j - c$, it is possible to define a function

$$\begin{aligned} P(x_j) &= \prod_{i=1}^{n}\left[x_j - c - (x_{j,i} - c)\right] f_{X_j}(x_j) \\ &= \prod_{i=1}^{n}(y_j - y_{j,i}) f_{X_j}(x_j) \end{aligned}, \quad (3)$$

where c is introduced to reduce the condition number and $f_{X_j}(x_j)$ denotes the probability density function of a random variable X_j.

Equation (3) satisfies

$$\int_{-\infty}^{\infty} P(x_j)(y_j)^i dx_j = 0;\ i = 0,1,\ldots,n-1. \quad (4)$$

If

$$r_{j,k} = \sum_{i_1=1}^{n}\sum_{i_2=1,\neq i_1}^{n}\cdots\sum_{i_k=1,\neq i_1,i_2,\ldots,i_{k-1}}^{n} y_{j,i_1}y_{j,i_2}\cdots y_{j,i_k}, \quad k=1,2,\ldots,n \quad (5)$$

Equation (4) yields a system of linear equations

$$\begin{bmatrix} \tilde{\mu}_{j,n-1} & -\tilde{\mu}_{j,n-2} & \tilde{\mu}_{j,n-3} & \cdots & (-1)^{n-1}\tilde{\mu}_{j,0} \\ \tilde{\mu}_{j,n} & -\tilde{\mu}_{j,n-1} & \tilde{\mu}_{j,n-2} & \cdots & (-1)^{n-1}\tilde{\mu}_{j,1} \\ \mu_{j,n+1} & -\mu_{j,n} & \mu_{j,n-1} & \cdots & (-1)^{n-1}\mu_{j,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{\mu}_{j,2n-2} & -\tilde{\mu}_{j,2n-3} & \tilde{\mu}_{j,2n-4} & \cdots & (-1)^{n-1}\tilde{\mu}_{j,n-1} \end{bmatrix} \begin{bmatrix} r_{j,1} \\ r_{j,2} \\ r_{j,3} \\ \vdots \\ r_{j,n} \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_{j,n} \\ \tilde{\mu}_{j,n+1} \\ \tilde{\mu}_{j,n+2} \\ \vdots \\ \tilde{\mu}_{j,2n-1} \end{bmatrix}, \quad (6)$$

where the coefficient matrix consists of known moments of input random variable X_j, given by

$$\tilde{\mu}_{j,i} = \int_{-\infty}^{\infty}(y_j)^i f_{X_j}(x_j)dx_j = \int_{-\infty}^{\infty}(x_j - c)^i f_{X_j}(x_j)dx_j. \quad (7)$$

After solving $r_{j,i}$ from Eq. (6), $y_{j,i}, i=1,\ldots,n$ can easily be obtained as the i th root of

$$y_j^n - r_{j,1}y_j^{n-1} + r_{j,2}y_j^{n-2} - \cdots + (-1)^n r_{j,n} = 0. \quad (8)$$

Finally, the integration nodes $x_{j,i}, i=1,\ldots,n$ can be obtained by adding $y_{j,i}$ to c.

The integration weights for each integration nodes can easily be determined by

$$\begin{aligned} w_{j,i} &= \frac{\int_{-\infty}^{\infty}\prod_{k=1,k\neq i}^{n}(y_j - y_{j,k})f_{X_j}(x_j)dx_j}{\prod_{k=1,k\neq i}^{n}(y_j - y_{j,k})} \\ &= \frac{\sum_{k=0}^{n-1}(-1)^k\tilde{\mu}_{j,n-k-1}q_{j,ik}}{\prod_{k=1,k\neq i}^{n}(y_j - y_{j,k})} \end{aligned}, \quad (9)$$

where $q_{j,i0} = 1$ and $q_{j,ik} = r_{j,k} - y_{j,i}q_{j,i(k-1)}$.

Compared to the MBQR, Eq. (6) with the singularity problem depends on the constant c, and the condition number of the system of linear equations also depends on the constant c. Given this, the last problem is then to determine the constant c that can reduce the condition number by as much as possible. To obtain the constant c, optimization is performed, in which an objective function is the condition number of a n by n matrix, as shown in Eq. (6). In addition, the statistical moments determined by Eq. (7) can easily be calculated, because they are calculated from the statistical raw moments which are already known.

2.3 Pearson system

In the Pearson system (Johnson *et al.*, 1995), the probability density function $f(x)$ is the solution of the differential equation

$$\frac{1}{f(x)}\frac{df(x)}{dx} = -\frac{\bar{x} + a}{c_0 + c_1\bar{x} + c_2\bar{x}^2}, \quad (10)$$

where $\bar{x}$ indicates $x - \mu$ and the coefficients a, c_0, c_1, and c_2 are determined by the first four statistical moments. The Pearson system is then classified as seven types according to the roots of $c_0 + c_1\bar{x} + c_2\bar{x}^2 = 0$.

The important factor related to the Pearson system is that the accuracy of the probability greatly depends on the values of skewness and kurtosis as well as on the accuracy of the calculated statistical moments. That is, although the calculated statistical moments are exact, if skewness and kurtosis are greatly deviated from those of a standard normal distribution, the accuracy of the probability calculated by the Pearson system can be reduced (Zhao and Ono, 2004).

3. Reliability based design optimization

Optimization has widely used to make more cost-effective production taking advantage of limited resources. Typical formulation of deterministic optimization is as follows:

$$\begin{aligned}&\text{Minmize } f(\mathbf{x})\\&\text{Subject to } g_j(\mathbf{x}) \le 0 \quad j=1,\ldots,m \quad ,\\&\qquad \underline{x}_i \le x_i \le \overline{x}_i \quad i=1,\ldots,n\end{aligned} \tag{11}$$

where $\mathbf{x}$ is a n-dimensional design vector, $\underline{x}_i$ is lower bound, and $\overline{x}_i$ is upper bound.

In contrast to deterministic optimization, RBDO can deal with the uncertainties of design variables and system parameters. Considering $g_j(\mathbf{X}) \le 0$ as a safety region, RBDO can be defined as

$$\begin{aligned}&\text{Minmize } f(\mathbf{d})\\&\text{Subject to } \Pr\left[g_j(\mathbf{X}) \ge 0\right] \le \Phi(-\beta_{t_j}) \quad j=1,\ldots,m ,\\&\qquad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U\end{aligned} \tag{12}$$

where $\mathbf{d} = \mu(\mathbf{X})$ is a n-dimensional design vector, $\mathbf{X}$ is a random variable vector, and the probabilistic constraint is described by the performance function $g_j(\mathbf{X})$ and its prescribed reliability target index β_{t_j}. Therefore, in RBDO, every time the optimizer needs the value of the probabilistic constraint, reliability analysis has to be done.

For reliability analysis, a fundamental problem is the computation of the multidimensional integral

$$P_f = \Pr\left[g(\mathbf{X}) \ge 0\right] = \int_{g(\mathbf{X})\ge 0} f(\mathbf{X}) d\mathbf{X} \ , \tag{13}$$

where $f(\mathbf{X})$ is the joint probability density function of $\mathbf{X}$, $g(\mathbf{X}) \ge 0$ represents the domain in which a failure occurs, and P_f is the probability of failure. To solve the multidimensional integral, approximated methods have been developed and one of them, FORM, has widely used for reliability analysis, especially for RBDO. As typical methods of RBDO based on the FORM, there are RIA and PMA. The two methods have a similar structure but there is a difference in sub-optimization.

4. A moment-based RBDO

4.1 Introduction

What makes a moment-based RBDO unique is that the moment method, instead of FORM, is used for the probabilistic constraint assessment, as depicted in Fig. 1. In the optimization process, the design variable vector, $\mathbf{d} = \mu(\mathbf{X})$, is continuously changed and, if the optimizer requires the values of the probabilistic constraints, the flow of the optimization enters the part with dotted lines. In this part, the integration nodes and weights of each random variable are first found, and then the first four statistical moments of each constraint are calculated. Finally, using the calculated statistical moments and the Pearson system, the probability for each constraint is calculated, and a corresponding reliability index is then calculated by

$$\beta = -\Phi^{-1}(P_f) \ , \tag{14}$$

where Φ^{-1} is the inverse cumulative distribution function of standard normal distribution. Then, the moment-based RBDO has the same structure as RIA and a difference is that a moment method, instead of FORM, is used for reliability analysis.

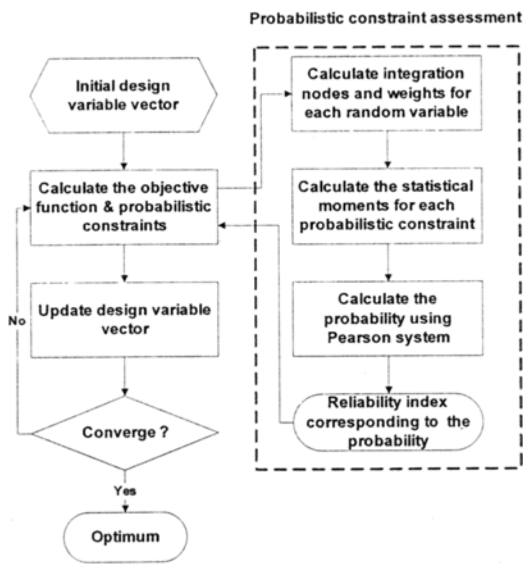

Fig. 1. A flowchart of a moment-based RBDO.

4.2 Considerations in developing a moment-based RBDO

One problem with a moment-based RBDO is that it can be impossible to calculate the sensitivity of a probabilistic constraint when the probability is numerically equal to 0 or 1. If the probability for a probabilistic constraint is numerically equal to 0, this indicates that a current design point is in the feasible region for the probabilistic constraint, and no problems arise. On the other hand, if the probability is numerically equal to 1, the current design point exists in the infeasible region for the probabilistic constraint. Generally, if the current design point exists in the infeasible region, most optimization algorithms initially attempt to bring the design point to the feasible region. During this process, sensitivity of the violated constraint is essentially needed. However, if the probability density function that is obtained from the Pearson system is bounded and the value of a random variable corresponding to the probability density function is larger than the upper bound, it is not possible to obtain sensitivity. For example, when the Type I distribution is used, the probability density function is bounded on each side, as in $[a,b]$. In this case, the probability for the value that is larger than the upper bound, b, is always 1 and even if a small variation is given, the probability remains unchanged, and the sensitivity eventually becomes 0. As a result, the optimizer cannot find a proper feasible region. Therefore, an initial design point must not violate any probabilistic constraints to a great extent.

To solve the problem, the optimum of deterministic optimization has to be assigned to the initial design point of the moment-based RBDO (Youn and Choi, 2004). This is done because deterministic optimization requires a comparatively small computation compared with RBDO and the optimum of deterministic optimization has approximately a 0.5 probability for an active probabilistic constraint.

5. Numerical examples

5.1 Comparison of MBQR and IMBQR

This example is introduced to confirm the performance of the IMBQR. For a comparison of MBQR and IMBQR, the condition number is compared for lognormal, Gumbel, and uniform distribution. The mean and the coefficient of variation are changed, and three nodes and weights are calculated.

Figure 2 shows the condition numbers for three non-normal distributions, in which the coefficient of variation is selected to be 0.01 when the mean is changed. From Fig. 2, the condition numbers of the IMBQR are much smaller than those of the MBQR. In addition, the condition number of the MBQR increases as the mean increases. Therefore, the MBQR cannot be used when the mean is very large.

Figure 3 shows the condition numbers for three non-normal distributions, in which the mean is selected to be 10 when the coefficient of variation is changed. From Fig. 3, similar results are obtained, but one difference is that the condition number of the MBQR increases as the coefficient of variance decreases. Therefore, the MBQR cannot be used when the coefficient of variance is very small.

In summary, since the IMBQR finds the minimum condition number through optimization, the condition number of the IMBQR is always smaller than that of the MBQR. Based on this advantage, the system of linear equations of the IMBQR is numerically more

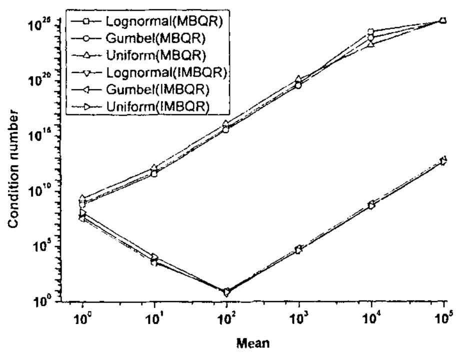

Fig. 2. Condition numbers for the variation of the mean.

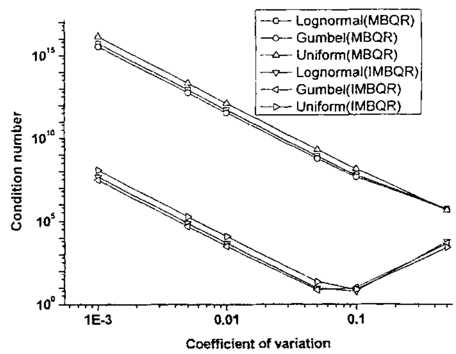

Fig. 3. Condition numbers for the variation of the coefficient of variance.

stable than that of the MBQR thus the IMBQR can calculate integration nodes and weights more accurately, even if the mean is very large or the coefficient of variance is very small.

5.2 Mathematical example

This numerical example (Youn and Choi, 2004) has 2 random variables and the formulation is as follows:

$$
\begin{aligned}
&\text{Minimize } f=(d_1+d_2)\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&\text{Subject to } \Pr\left[g_1 \equiv 1-\frac{x_1^2x_2}{20}\geq 0\right]\leq \Phi(-\beta_1)\\
&\Pr\left[g_2 \equiv 1-\frac{(x_1+x_2-5)^2}{30} -\frac{(x_1-x_2-12)^2}{120}\geq 0\right]\leq \Phi(-\beta_2)\\
&\Pr\left[g_3 \equiv 1-\frac{80}{(x_1^2+8x_2+5)}\geq 0\right]\leq \Phi(-\beta_3)\\
&0\leq d_i \leq 10 \quad \text{for } i=1,2\\
&x_i \sim N(d_i, 0.3) \quad \text{for } i=1,2
\end{aligned}
\qquad (15)
$$

An initial design point is the optimum of deterministic optimization (3.1107, 2.0609).

RBDO results are summarized in Tables 1-3, where four different distributions are used for three different target reliability indexes and three-node integration is used for a moment method. In Tables 1-3, 'RBDO (Rosenblatt)' indicates a moment-based RBDO, in which integration nodes are determined by the inverse Rosenblatt transformation, and 'RBDO (IMBQR)' denotes a moment-based method, in which integration

Table 1. Results of different approaches in the case of 1-sigma RBDO.

Distribution	Approach	Objective	d_1	d_2	β^1_{MCS}	β^2_{MCS}
Normal	PMA	5.689	3.183	2.506	0.989	1.111
	RBDO (Rosenblatt)	5.684	3.204	2.481	1.001	1.006
	RBDO (IMBQR)	5.684	3.204	2.481	1.001	1.006
Lognormal	PMA	5.698	3.191	2.507	1.024	1.117
	RBDO (Rosenblatt)	5.683	3.193	2.490	1.018	1.028
	RBDO (IMBQR)	5.680	3.203	2.477	1.003	1.004
Gumbel	PMA	5.689	3.202	2.486	1.105	1.116
	RBDO (Rosenblatt)	5.681	3.197	2.484	1.081	1.119
	RBDO (IMBQR)	5.652	3.199	2.453	1.009	1.014
Uniform	PMA	5.821	3.210	2.611	1.194	1.260
	RBDO (Rosenblatt)	5.641	3.193	2.448	0.842	0.850
	RBDO (IMBQR)	5.728	3.211	2.518	1.024	1.000

Table 2. Results of different approaches in the case of 2-sigma RBDO.

Distribution	Approach	Objective	d_1	d_2	β^1_{MCS}	β^2_{MCS}
Normal	PMA	6.202	3.297	2.905	1.993	2.074
	RBDO (Rosenblatt)	6.194	3.312	2.882	1.992	1.989
	RBDO (IMBQR)	6.194	3.312	2.882	1.992	1.991
Lognormal	PMA	6.156	3.290	2.867	2.030	2.082
	RBDO (Rosenblatt)	6.113	3.289	2.824	1.944	1.944
	RBDO (IMBQR)	6.132	3.294	2.838	1.990	1.986
Gumbel	PMA	6.022	3.243	2.779	2.104	2.109
	RBDO (Rosenblatt)	5.875	3.200	2.675	1.597	1.781
	RBDO (IMBQR)	5.964	3.237	2.728	1.923	1.915
Uniform	PMA	6.196	3.302	2.894	2.275	2.308
	RBDO (Rosenblatt)	6.107	3.289	2.818	1.959	1.884
	RBDO (IMBQR)	6.112	3.281	2.832	1.959	1.971

Table 3. Results of different approaches in the case of 3-sigma RBDO.

Distribution	Approach	Objective	d_1	d_2	β^1_{MCS}	β^2_{MCS}
Normal	PMA	6.731	3.441	3.290	2.993	3.063
	RBDO (Rosenblatt)	6.703	3.446	3.257	2.956	2.954
	RBDO (IMBQR)	6.703	3.446	3.257	2.955	2.954
Lognormal	PMA	6.597	3.404	3.192	3.022	3.084
	RBDO (Rosenblatt)	6.449	3.376	3.073	2.707	2.680
	RBDO (IMBQR)	6.551	3.398	3.153	2.934	2.947
Gumbel	PMA	6.298	3.285	3.013	3.095	3.127
	RBDO (Rosenblatt)	6.212	2.865	3.348	1.554	Infinite
	RBDO (IMBQR)	6.188	3.261	2.927	2.662	2.725
Uniform	PMA	6.307	3.335	2.972	3.006	3.102
	RBDO (Rosenblatt)	6.575	3.408	3.167	Infinite	Infinite
	RBDO (IMBQR)	6.262	3.324	2.938	2.649	2.633

nodes and weights are calculated by the IMBQR. In addition, the sixth and seventh columns are the reliability index calculated by MCS at the optimum design. From Tables 1-3, it is shown that the optimums are similar when the random variables are normal distribution or the target reliability index is 1 or 2. However, when the target reliability index is 3 and random variables are Gumbel or uniform distributions, there is a difference between RBDO (Rosenblatt) and the other methods in the optimum.

To check the accuracy of evaluating the probabilistic constraints, Monte Carlo simulation (MCS) with a sample size of ten million is used for active probabilistic constraints at the optimum design and the error is defined as follows:

$$\left|\frac{(\beta^{i}_{\mathrm{MCS}}-\beta_t)}{\beta_t}\right|\times 100\%\,, \tag{16}$$

where β^{i}_{MCS} is the reliability of the ith performance function by MCS for the target reliability β_t. The results are displayed in Figs. 4-6 for the first probabilistic constraint. In Figs. 4-6, the errors are very small for all the methods when the random variables are normal distributions. For non-normal distributions, the errors of RBDO (IMBQR) are comparatively smaller than the other methods when the target reliability index is equal to 1 or 2. However, the errors of the moment-based method are larger than those of PMA when the target reliability index is equal to 3, especially when RBDO (Rosenblatt) is used. For this, there are two reasons. The first reason is that errors occur in the computation of the statistical moments, as shown in Table 4, where the statistical moments are calculated in the initial design point for the first probabilistic constraint. From Table 4, it is observed that the moment method that uses the inverse Rosenblatt transformation has large errors in skewness and kurtosis of non-normal distributions, while the method that uses the IMBQR has only small errors in kurtosis. This makes the errors with the method that uses the inverse Rosenblatt transformation much larger. The second reason is that the Pearson system can be inaccurate in some cases. It is generally known that the more skewness and kurtosis are deviated from those of a standard normal distribution, the larger the errors in probability are even if the statistical moments are exactly calculated. This is because the Pearson system uses only four statistical moments.

In this example, the results of the MBQR were omitted because the results of the MBQR are identical to those of the IMBQR. The reason why the results are the same is that the means are relatively small and the coefficient of variances are approximately 0.1. In this case, the condition number can be expected to be smaller than 10^{10}, as shown in Figs. 2 and 3, and the real maximum condition numbers for each distribution are displayed in Table 5. Although a numerical

Table 4. Comparison of four statistical moments for the first probabilistic constraint at the initial design point.

Distribution	Approach	Mean	Standard	Skewness	Kurtosis
Normal	Rosenblatt	0.0064	0.2437	0.4797	3.3227
	IMBQR	0.0064	0.2437	0.4796	3.3227
	MCS	0.0063	0.2437	0.4792	3.3592
Lognormal	Rosenblatt	0.0064	0.2459	0.7275	3.7057
	IMBQR	0.0064	0.2459	0.7473	3.9671
	MCS	0.0065	0.2460	0.7505	4.0212
Gumbel	Rosenblatt	0.0065	0.2504	1.1646	4.4864
	IMBQR	0.0064	0.2525	1.4727	7.2715
	MCS	0.0064	0.2525	1.4706	7.3933
Uniform	Rosenblatt	0.0049	0.2230	0.4401	3.2719
	IMBQR	0.0064	0.2435	0.3889	2.4987
	MCS	0.0064	0.2435	0.3893	2.5027

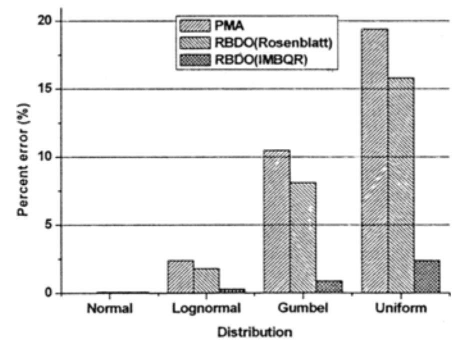

Fig. 4. Errors in the first probabilistic constraint in the case of 1-sigma RBDO.

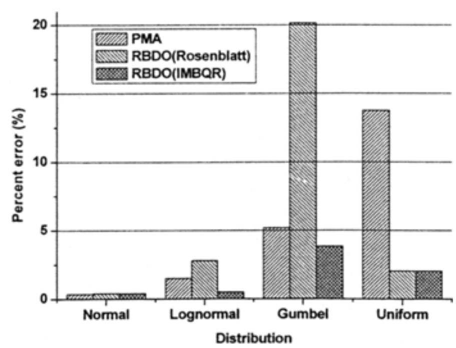

Fig. 5. Errors in the first probabilistic constraint in the case of 2-sigma RBDO.

Table 5. Maximum condition numbers of the mathematical example in the process of optimization.

	Normal	Lognormal	Gumbel	Uniform
MBQR	3,347,840	2,973,640	50,150,300	7,331,090
IMBQR	62	62	46	157

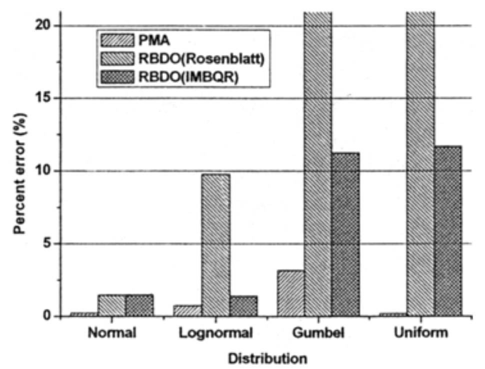

Fig. 6. Errors in the first probabilistic constraint in the case of 3-sigma RBDO.

problem does not occur in the MBQR, it can be seen that the condition numbers of the IMBQR are much smaller than those of the MBQR. Thus, using the IMBQR is more proper, especially when integration nodes and weights are calculated many times such as optimization.

In the number of function calls, the moment-based RBDO requires 9 computations for reliability analysis when three-node integration is used. In this case, the moment method is more efficient than PMA, which uses optimization for reliability analysis. However, it can be seen that the number of function calls increases very fast as the number of random variables increases.

5.3 A welded beam

The second numerical example (Lee and Lee, 2005) is a welded beam structure as shown in Fig. 7. It has 4 random variables and 5 probabilistic constraints. The objective function is the welding cost and constraints are imposed on geometry, the maximum possible stress, and the tip deflection. The design variables are displayed in Fig. 7, and each design variable follows a statistically independent normal distribution. The system parameters and the variances of the random variables are given in Table 6. The description of the optimization problem is as follows:

Table 6. Data for the welded beam problem.

z_1	2.6688×10^4 [N]	z_6	9.377×10 [MPa]
z_2	3.556×10^2 [mm]	z_7	2.0685×10^2 [MPa]
z_3	2.0685×10^5 [MPa]	c_1	6.74135×10^{-5} [\$/mm^3]
z_4	8.274×10^4 [MPa]	c_2	2.93585×10^{-6} [\$/mm^3]
z_5	6.35 [mm]		
σ_{xi}^2	$\begin{bmatrix}2.8674\times10^{-2} & 2.8674\times10^{-2} & 1.1470\times10^{-4} & 1.1470\times10^{-4}\end{bmatrix}$		

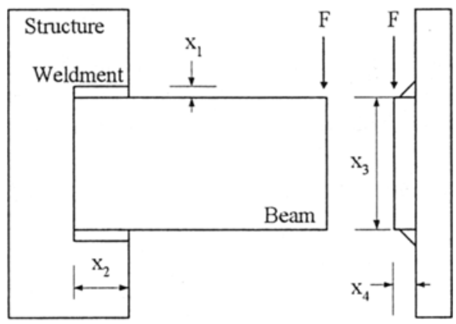

Fig. 7. A welded beam structure.

$$\begin{aligned}
&\textit{find}\ \mathbf{x}=(x_1,x_2,x_3,x_4)\\
&\textit{minimize}\ \ f(\mathbf{x},\mathbf{z})=c_1x_1^2x_2+c_2x_3x_4(z_2+x_2)\\
&\textit{subject to}\ \ \Pr\left[g_i(x)\ge 0\right]\le\Phi(-\beta_i)\ \ ,\ i=1,\cdots,5\\
&\textit{where}\\
&\quad g_1(\mathbf{x})=\tau(\mathbf{x},\mathbf{z})/z_6-1,\quad g_2(\mathbf{x})=\sigma(\mathbf{x},\mathbf{z})/z_7-1\\
&\quad g_3(\mathbf{x})=x_1/x_4-1,\quad g_4(\mathbf{x})=\delta(\mathbf{x},\mathbf{z})/z_5-1,\\
&\quad g_5(\mathbf{x})=1-P_c(\mathbf{x},\mathbf{z})/z_1\\
&\quad \beta_1=\beta_2=\cdots=\beta_5=3.0\\
&\quad 3.175\le x_1\le 50.8\ ,\ 0\le x_2\le 254\ ,\ \ 0\le x_3\le 254\ ,\\
&\quad 0\le x_4\le 50.8
\end{aligned}$$

$$\tau(\mathbf{x},\mathbf{z})=\left\{\left\{(t(\mathbf{x},\mathbf{z})^2+2t(\mathbf{x},\mathbf{z})tt(\mathbf{x},\mathbf{z})(x_2/2R(\mathbf{x}))+tt(\mathbf{x},\mathbf{z})^2\right\}^{1/2}\right. \tag{17}$$

$$t(\mathbf{x},\mathbf{z})=z_1/\sqrt{2}x_1x_2$$

$$tt(\mathbf{x},\mathbf{z})=M(\mathbf{x},\mathbf{z})R(\mathbf{x})/J(\mathbf{x})$$

$$M(\mathbf{x},\mathbf{z})=z_1\left(z_2+\frac{x_2}{2}\right),$$

$$R(\mathbf{x})=\frac{\sqrt{x_2^2+(x_1+x_3)^2}}{2}$$

$$J(\mathbf{x})=\sqrt{2}x_1x_2\left\{x_2^2/12+(x_1+x_3)^2/4\right\}$$

$$\sigma(\mathbf{x},\mathbf{z})=6z_1z_2/x_3^2x_4$$

$$\delta(\mathbf{x},\mathbf{z})=4z_1z_2^3/z_3x_3^3x_4$$

$$P_c(\mathbf{x},\mathbf{z})=\frac{4.013x_3x_4^3\sqrt{z_3z_4}}{6z_2^2}\left(1-\frac{x_3}{4z_2}\sqrt{\frac{z_3}{z_4}}\right)$$

Table 7. Summary of the optimization results for the welded beam problem.

	Objective	Design variables	Number of FE
Deterministic Optimization	2.38	(6.212, 157.5, 210.6, 6.207)	
PMA	2.59	(5.750, 219.8, 210.7, 6.260)	79,281
RBDO (Rosenblatt)	2.59	(5.730, 200.6, 210.6, 6.239)	32,805
RBDO (IMBQR)	2.59	(5.730, 200.6, 210.6, 6.239)	32,805
RBDO (MBQR)		Fail	

The optimization results are summarized in Table 7, where target reliability indexes are 3 and the result of PMA comes from Lee and Lee (2005). From Table 7, the results of PMA and the moment-based RBDO are nearly identical, but the moment-based method is more efficient than PMA, where the moment-based method requires 81 computations for reliability analysis. In the results, it is interesting to note that 'RBDO (MBQR)' cannot find the optimum because the condition number is so large that integration nodes and weights cannot be exactly calculated. In this example, the maximum condition number of the MBQR is 1.8×10^{21} while that of the IMBQR is 3.8×10^{7}. The reason why the condition number is very large is that the means of several random variables are comparatively large and the coefficient of variance is very small because of a very small variance, as shown in Table 6. Therefore, the MBQR cannot be applied to an example with large means or small coefficient of variances.

5.4 Discussions

The application of a moment method to RBDO was investigated. In this process, since some probability density functions that were obtained from the Pearson system had bounds, a case in which sensitivity analysis of probabilistic constraints was impossible occurred when the value of a random variable existed outside the bounds, especially when the value was larger than the upper bound. In this case, the design point existed in the infeasible region, thus it was impossible to find the optimum. Therefore, the initial design point of a moment-based RBDO must not excessively violate the probabilistic constraints. As a solution for this case, the optimum of deterministic optimization was used as an initial design point.

In the process of the moment-based RBDO, integration nodes and weights for the moment method should be calculated. Furthermore, because they are recalculated for each probabilistic constraint every time the design point is changed, it is important to calculate them accurately and reliably. When the inverse Rosenblatt transformation was used, the computation was very simple because integration nodes were only changed, but the accuracy of the statistical moments could be reduced, especially when non-normal distributions were dealt with. On the other hand, when the MBQR was used, the computation was comparatively complex because of solving a system of linear equations. Here, however, the accuracy of the MBQR was superior to that of the inverse Rosenblatt transformation.

Nonetheless, since the system of linear equations of the MBQR may be singular when the mean of a random variable was very large or the coefficient of variance of a random variable was very small, using the MBQR is restricted in application of RBDO. To overcome this drawback, the IMBQR was proposed, and it was confirmed through numerical examples that the condition number of the IMBQR was much smaller than that of the MBQR. Furthermore, for a welded beam example, the IMBQR could obtain the optimum while the MBQR failed to find the optimum.

The accuracy of the moment-based RBDO depends on the accuracy of the calculated statistical moments and the values of skewness and kurtosis. The results showed that the moment-based RBDO using the IMBQR was relatively accurate when the target reliability index was 1 or 2, while PMA was accurate when the target reliability index was 3. This is because the Pearson system uses only four statistical moments. For this reason, the accuracy of a probability density function, which is obtained from the Pearson system, can be low, especially when the probability is determined in the tail of the probability density function. Therefore, the Pearson system should be used with caution when the target reliability index is larger than 2. On the other hand, the errors in PMA decrease as the target reliability index is getting larger because the joint probability density function to be integrated is exponentially decayed toward the tail.

Finally, a disadvantage with the moment-based RBDO is that the number of function calls increases very fast as the number of random variables increases. It is estimated that the problem will be solved by using metamodels such as Response surface method, Kriging, and Radial basis function.

6. Concluding remarks

Unlike previous research regarding RBDO, a moment-based RBDO was investigated. It has an advantage in that it does not require optimization for reliability analysis. In this process, an improved moment-based quadrature rule, which improved numerical stability, was proposed. Applying the moment-based RBDO to examples with different target reliability indexes and distributions, it could be confirmed that the moment-based RBDO using the IMBQR was considerably accurate when the target reliability index was 1 or 2 irrespective of the type of distributions. In addition, the problem that the number of function calls increases very fast as the number of the random variables increases is expected to be overcome through the use of metamoldels, in the future.

Acknowledgements

This research was supported by The Center of Innovative Design Optimization Technology, Korea Science and Engineering Foundation.

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